**AN ALTERNATIVE IMPLEMENTATION OF INDICATOR KRIGING**

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## Abstract

Indicator kriging (IK) provides a non-parametric distribution estimated directly at fixed thresholds by considering indicator transforms of conditioning data in the form of cumulative distribution functions with 0/1 step functions. To adequately estimate all distributions, IK is commonly implemented with numerous thresholds using large numbers of conditioning data, resulting in significant order relation violations. An alternative implementation of IK is proposed, involving the direct coding of the conditioning data in the form of probability distribution functions with 0/1 step functions, and the dynamic relocation of thresholds to the conditioning data values. Improved local accuracy and precision, and recoverable reserve estimates were obtained from the proposed implementation of IK when compared to the traditional IK method in an artificial case study. Computer programs *dtik3d* and *dtpik* are provided to implement the proposed dynamic threshold IK technique.

**Key words:** Dynamic thresholds, Conditional distributions, Local accuracy and precision

## 1.0. Introduction

The advantage of IK over distribution based estimators such as ordinary kriging is that IK provides a non-parametric distribution, estimated directly at various thresholds. However, the selection of these fixed thresholds based on the global distribution involves conflicting goals (Deutsch, Journel, 1997). One of the most important considerations is the selection of sufficient thresholds to describe adequately the conditional cumulative distribution function (ccdf) for all locations.

For strongly positively skewed populations, such as precious metal deposits, resolution of the ccdf is lost if too few thresholds are used. Fig. 1 shows the ccdf's for 10 locations estimated by IK using both 5 and 12 thresholds. In this diagram, when 5 thresholds are used for IK the important upper parts of the distribution are poorly resolved (compare the right-hand diagrams in Fig. 1). For the IK implementation with 5 thresholds, there was an average 2.3 order relation problems per location with an average magnitude of 0.042.

The use of large numbers of thresholds, especially in the important parts of the distribution, to resolve adequately all ccdf's is a common IK practice. However, if numerous thresholds are used, with each threshold being independently estimated by IK, the resulting distribution commonly violates the axioms of a cdf, often leading to significant order relation corrections. When using 12 thresholds in the previous example, there was an average of 7.4 order relation problems per location with an average magnitude of 0.033. These order relation violations result from the non-convexity of the kriging algorithm, lack of conditioning data in some specific classes (Deutsch, Journel, 1997), and the independent modelling of indicator variograms at the various threshold values (Journel, Posa, 1990). In most practical implementations, large numbers of indicator thresholds are also accompanied by a significant increase in the number of conditioning data and, consequently, CPU time. Clearly, an ideal implementation of IK is one that rapidly provides sufficient ccdf resolution with no order relation violations.

This paper explores an alternative IK implementation involving the dynamic relocation of thresholds to the conditioning data grade values, and proposes a new approach to constructing the ccdf. This implementation of IK involves the direct coding of the experimental data in the form of probability distribution functions (pdf) with 0/1 step functions, then using the conditional expectation of these indicator random variables at unsampled locations to generate the ccdf. The practical application of the proposed methodology is demonstrated and contrasted to the traditional implementation of IK in an artificial recoverable reserves case study.

## 2.0 Indicator kriging

This section briefly reviews the theoretical considerations for IK. A thorough presentation of the theory and practice of indicator geostatistics is available in textbooks such as Deutsch, Journel (1997) and Goovaerts (1997).

### 2.1 Traditional IK

The basic approach of probabilistic geostatistics, such as IK, is to turn any unsampled value $z(x)$ into a random variable $Z(x)$. The probability distribution function (pdf) of $Z(x)$ characterises the uncertainty about the unknown true value $z(x)$. The cdf of $Z(x)$ is denoted:

$$F(x;z) = \Pr\{Z(x) \le z\} \tag{1}$$

where $F(x;z) \le F(x;z') \ \forall \ z \le z'$.

In practice, a set of *n* conditioning data $\{z(x_\alpha), \alpha = 1,\dots,n\}$ is used:

$$F(x;z|(n)) = \Pr\{Z(x) \le z|(n)\} \tag{2}$$

Traditional indicator geostatistics relies on the direct coding of experimental data in the form of cdf's with 0/1 step functions, *i.e.* indicator variables defined:

$$i(x_\alpha;z) = \begin{Bmatrix} 1 & \text{if } z(x_\alpha) \le z; \\ 0 & \text{otherwise} \end{Bmatrix} \tag{3}$$

In indicator kriging, the ccdf is discretised into *K+1* classes using *K* thresholds $\{z_k, k = 1,\dots,K\}$ and the probability bounds, then estimated independently at each threshold value by its conditional expectation (Journel, 1983):

$$\begin{aligned} E\{I(x;z_k|(n))\} &= 1 * \Pr\{Z(x) \le z_k|(n)\} + 0 * \Pr\{Z(x) > z_k|(n)\} \\ &= \Pr\{Z(x) \le z_k|(n)\} \\ &= F(x;z_k|(n)) \qquad k = 1,\dots,K \end{aligned} \tag{4}$$

where $E\{I(x;z_k|(n))\}$ is the conditional expectation of the random variable $I(x;z_k)$. Note that there is no constraint for $F(x;z_{k-1}|(n)) \le F(x;z_k|(n))$.

## 2.2 Alternate implementation of IK

An alternative IK implementation, involving the dynamic relocation of the threshold values $z_k$ to the *z*-values of the *n* conditioning data at each unknown location *x* was suggested, but not described, by Deutsch, Lewis (1992). Chu (1994) proposed modifications to the traditional implementation of IK to rapidly account for large numbers of threshold values that effectively equates to this alternative IK approach. However, CPU effective implementation relied on partial ccdf determination and, consequently, non-correction of order relation violations. This section describes the approach suggested by Deutsch, Lewis (1992).

To enable construction of the ccdf (see Appendix A), IK with dynamic threshold selection (DTIK) requires experimental data to be coded in the form of pdf's with 0/1 step functions, *i.e.* indicator variables defined:

$$j(x_\alpha;z) = \begin{Bmatrix} 1 & \text{if } z(x_\alpha) = z; \\ 0 & \text{otherwise} \end{Bmatrix} \tag{5}$$

Then, the conditional probability distribution function (cpdf) for each *z*-value corresponding to the conditioning data is estimated by its conditional expectation:

$$\begin{aligned} E\{J(x;z_\alpha|(n))\} &= 1 * \Pr\{Z(x) = z_\alpha|(n)\} + 0 * \Pr\{Z(x) \neq z_\alpha|(n)\} \\ &= \Pr\{Z(x) = z_\alpha|(n)\} \\ &= p^*(x;z_\alpha|(n)) \qquad\qquad \alpha = 1,\dots,n \end{aligned} \tag{6}$$

where $E\{J(x;z_\alpha|(n))\}$ is the conditional expectation of the random variable $J(x;z_\alpha)$.

Note that, if the same variogram is used to estimate all *n* probabilities, then they sum to one and are equivalent to the kriging weights $\lambda(x_\alpha)$ returned by ordinary kriging (Rao, Journel, 1997), *i.e.* $p^*(x;z_\alpha|(n)) = \lambda(x_\alpha)$. However, the non-convex property of kriging does not ensure the absence of non-physical probabilities. A new method of adjusting negative weights to satisfy the axioms of a pdf that preserves the relative magnitude between sample kriging weights is proposed in Appendix B.

A problem with DTIK is that it calls for the definition of a variogram at an infinite number of threshold values, corresponding to each sample grade $z_\alpha$. In addition, at each threshold when using the indicator transform in Eq. (5) only one datum is coded 1 and $N$-1 are coded 0, where $N$ is the total number of samples. Experimental variograms calculated from these indicator datasets display pure nugget effects. Solutions to this problem include defining a finite set of variograms $\{\gamma(z_k), k = 1,...K\}$ using data coded as the traditional indicator transform in Eq. (3), then using either a variogram interpolation approach (Chu, 1994), or a variogram class approach where $\gamma(z_\alpha) = \gamma(z_k) \forall\, z_{k-1} \leq z_\alpha < z_k$. The respective solutions call for the solving of $n$ and $K$ kriging systems.

Fig. 2 shows the ccdf's using DTIK for the same 10 locations shown in Fig. 1. In this diagram, the important upper parts of the distribution are relatively well resolved when only 12 samples are used for kriging (compare the right-hand diagrams in Figs 1 and 2). The use of additional samples for DTIK provides little additional benefit in resolving the ccdf. This is evident by comparing the top and bottom diagrams in Fig. 2, for which 12 and 24 conditioning data were used for DTIK respectively.

## 3.0 Case study

An artificial case study using the GSLIB dataset is developed to evaluate the proposed methodology against traditional indicator geostatistics. Fig. 3 shows the spatial distribution of 50 x 50 measurements, and hereafter considered as the reference dataset. In this example, the measurements are considered to be from a precious metal deposit, thus the high values are more economically important.

One hundred and fifty locations were drawn randomly on twenty occasions to form 20 sample datasets, the first of which is shown in Fig. 3. The reference and sampled statistics differed somewhat, shown in Fig. 4. However, on average, the sample datasets represent the true distribution, shown in Fig. 5, where the sample datasets fluctuate closely about the 45 degree line of Q-Q plots.

Indicator variogram model parameters for the deciles of a declustered sample population of 140 samples are listed in Table 1. All models consist of a nugget effect and either one or two spherical structures. A further 3 thresholds were defined at the 95$^{th}$, 97.5$^{th}$, and 99$^{th}$ percentiles and assigned the indicator variogram model of the 9$^{th}$ decile. The additional thresholds were designed to accurately model the important higher grade parts of the conditional distributions.

The GSLIB program *ik3d* was used to generate ccdf's from all 20 datasets for 5 x 5 blocks using: 1) 5 thresholds (1$^{st}$, 3$^{rd}$, 5$^{th}$, 7$^{th}$, and 9$^{th}$ deciles) and 12 samples, hereafter referred to as IK-1; and 2) all 12 thresholds and 24 samples (IK-2). A modified version of *ik3d* (*dtik3d*) generated an additional two sets of ccdf's from all 20 datasets for 5 x 5 blocks with the DTIK method using 12 samples (DTIK-1), and 24 samples (DTIK-2). Both DTIK-1 and DTIK-2 applied the 5 variograms used in IK-1 as class variograms, thus, only 5 kriging systems were solved per location. Negative weights were adjusted to satisfy the axioms of a pdf using the methodology shown in Appendix B. In all four IK implementations, the same search neighbourhood and search strategy was employed.

Some statistical characteristics of the models generated from the various IK implementations are shown in Table 2. In this table, the mean block grade for all IK implementations are close to the true block mean of 2.58 units, however, the variance of the block grades differ significantly. For example, IK-1, IK-2, DTIK-1, and DTIK-2 returned block variances of 4.00, 4.12, 5.87, and 3.87 units$^2$ respectively, significantly below the true block variance of 9.24 units$^2$. Thus, if only the block mean grades are considered, significant smoothing is present in all of the models.

Local accuracy, precision, and goodness measures (Deutsch, 1997) were determined for all IK models. These three measures are shown in Figure 6, local accuracy plots for the first IK model from each estimation method. In Figure 6, the plot for DTIK-1 is closest to the ideal 45$^o$ line suggesting that this implementation of IK produces the best probabilistic model. The other three models deviate somewhat from this 45$^o$ line, reflected by their inferior local accuracy statistics. Similar characteristics were noted for the IK models constructed from the remaining 19 sample datasets.

Table 2 includes the average co-efficient of variation for the conditional distributions of the four implementations. The greater the co-efficient of variation the greater the spread of the conditional distribution relative to its mean. The average co-efficient of variation of the conditional distributions is slightly higher than the true value for DTIK-1 and significantly greater than the true value for IK-1, IK-2, and DTIK-2. These higher values are an indication of smoothing of the conditional distributions, thus, these models may not be appropriate for estimating proportions above a cut-off.

Recoverable reserves for a 2.5 unit cut-off were obtained from the IK-1 and IK-2 models using the GSLIB program *postik* and suitable interpolation and upper tail extrapolations. As the dynamic threshold IK method produces varying numbers of ccdf points at each location a modified version of *postik* (*dtpik*) was used to determine recoverable reserves for DTIK-1 and DTIK-2 models. The variance reduction factor used was 1, thus, the results represent the recoverable reserves for the reference 1 x 1 support.

Histograms of the recoverable tonnage error for the various IK implementations are shown in Fig. 7. In this diagram a positive value represents overestimation of recoverable tonnes relative to the reference dataset and vice versa. Both of the traditional implementations of IK, on average, overestimate the recoverable tonnage by more than 5% at this cut-off, shown in Figs 7A and 7B. The use of additional thresholds, to better define the ccdf slightly improves the average recoverable tonnage error from 5.9 to 5.3%. DTIK appears to be unbiased with average recoverable tonnage errors <1%, shown in Figs 7C and 7D.

It is possible that four factors are biasing the recoverable reserve results returned by IK-1 and IK-2:

1. Screening of data is more pronounced for lower nugget effects, thus, low grade classes (<8$^{th}$ decile) are significantly influenced by negative kriging weights.
2. Linear interpolation between fixed thresholds on the ccdf provides a within-class uniform distribution, which may not be appropriate for high grade classes in strongly positively skewed deposits. Solutions include linear interpolation between tabulated values within each class

(Deutsch, Journel, 1997), and using a threshold equivalent to the cut-off grade. In practice, the tabulated values are the global distribution, which might not be locally representative.

3. The correction of significant order relation violations, resulting from sudden changes in nugget effect and spatial variance at the 8$^{th}$ and 9$^{th}$ deciles, may not be appropriate.

The increased average block co-efficient of variation values from IK-1 to IK-2 and from DTIK-1 to DTIK-2, shown in Table 2, are related to ccdf's in low grade areas being informed by additional high grade samples, often at some distance. This smoothing effect, enhanced by the 90% nugget effect above the 9$^{th}$ decile, is evidenced by the greater spread of recoverable tonnage results for IK-2 and DTIK-2 than for IK-1 and DTIK-1, shown in Fig. 7. The smoothing impact is more pronounced for DTIK as the local ccdf's that result from DTIK tend toward the global cdf as more and more samples are used as conditioning data.

Histograms of the recoverable metal error for the various IK implementations are shown in Fig. 8. In this diagram a positive value represents overestimation of recoverable metal relative to the reference dataset and vice versa. In Fig. 8A, the traditional implementation of IK with 5 cut-offs, on average, underestimates the recoverable metal by 3.5%. These results are sensitive to the hyperbolic upper tail extrapolation parameter $\omega$. The use of additional thresholds in IK-2, on average, underestimates the recoverable metal by 4.7%. Overestimation of tonnes was a characteristic of both traditional IK implementations, thus, significant underestimation of grade is present. When dynamic thresholds are used (DTIK-1 and DTIK-2) the recoverable metal, on average, is close to the true value, shown in Figs 8C and 8D.

## 4.0 Conclusions

The traditional implementation of IK provides a non-parametric distribution estimated directly at fixed user-defined thresholds. IK models in an artificial case study using this approach were sensitive to the selection of these fixed thresholds and the extrapolation from discrete conditional probabilities to the probability bounds. Irrespective of the number of thresholds, the conditional distributions were biased, resulting in poor recoverable reserve estimates.

An alternate implementation of IK, where the thresholds are dynamically relocated to the conditioning data values, was proposed. In the case study, DTIK using few conditioning data resulted in the best probabilistic model and unbiased recoverable reserve estimates. As additional conditioning information is used for DTIK, the local conditional distributions are significantly smoothed as they tend towards the global cdf. Thus, locally accurate DTIK models relied on using small numbers of conditioning data, an appropriate method of ccdf construction, and the absence of negative kriging weights.

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# Appendix A – Constructing cdf's from conditional probabilities

Consider the conditional probabilities returned by kriging indicator transforms of the eight data listed in Table A1 and graphed in Fig. A1. The mean and variance of the distribution are 1.36 ppm and 1.23 $ppm^2$ respectively. These conditioning data belong to a global population with minimum and maximum grade values of 0.0 and 8.0 ppm respectively. If these conditional probabilities are used to build a ccdf, then it appears as an inappropriate step function bounded by the minimum and maximum *z*-values of the conditioning data, shown in Fig. A2.

Table A1. Conditional probabilities for 8 conditioning data.

| Sample | Grade | Conditional Probability | *z*-range of influence | |
|---|---|---|---|---|
| | | | lower | upper |
| 1 | 0.1 | 0.10 | 0.00 | 0.16 |
| 2 | 0.2 | 0.05 | 0.12 | 0.31 |
| 3 | 0.3 | 0.15 | 0.19 | 0.43 |
| 4 | 0.4 | 0.20 | 0.28 | 0.60 |
| 5 | 1.8 | 0.10 | 1.40 | 2.40 |
| 6 | 2.0 | 0.05 | 1.55 | 2.65 |
| 7 | 2.3 | 0.10 | 1.70 | 3.00 |
| 8 | 2.8 | 0.25 | 2.15 | 3.55 |

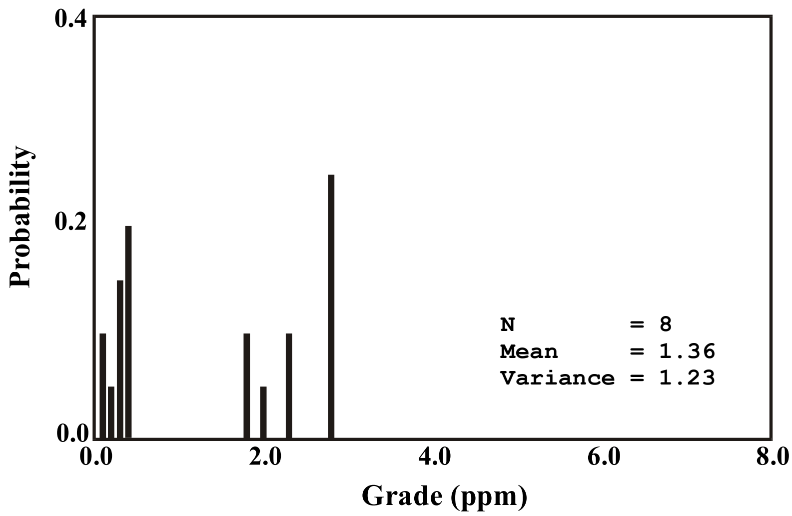


Figure A1. Probability distribution of conditioning data listed in Table A1.

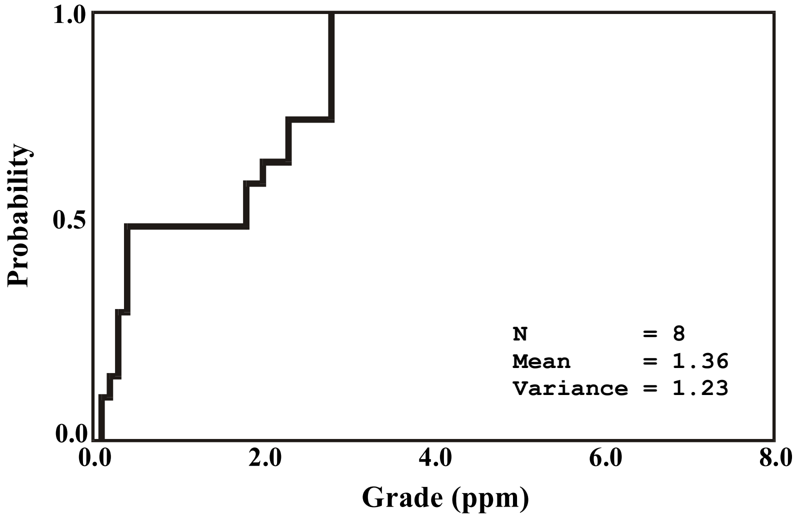


Figure A2. Cumulative distribution of conditioning data listed in Table A1.

One possible solution is to construct a ccdf using:

1. the discrete mid-points of the vertical steps, calculated as:

$$F(x;z_\alpha|(n)) = F(x;z_{\alpha-1}|(n)) + 0.5\left[p^*(x;z_{\alpha-1}|(n)) + p^*(x;z_\alpha|(n))\right], \text{ and} \quad \text{(A1)}$$

2. discrete points on the horizontal steps inversely proportional to the conditional probabilities of adjacent samples:

$$F(x;z_{\alpha'}|(n)) = F(x;z_\alpha|(n)) + 0.5\left[p^*(x;z_\alpha|(n))\right] \qquad \text{(A2)}$$

where $z_{\alpha'} = \left[(z_\alpha \cdot p^*(x;z_\alpha|(n))) + (z_{\alpha+1} \cdot p^*(x;z_{\alpha+1}|(n)))\right] / (p^*(x;z_\alpha|(n)) + p^*(x;z_{\alpha+1}|(n)))$.

Note that, as each class of $F(x;z_\alpha|(n))$ is informed, order relation violations are not possible if the kriging system returns only positive weights.

The ccdf constructed using discrete points determined from Eqs A1 and A2, and the previous example bounded by the population limits of 0.0 and 8.0 ppm, is shown in Fig. A3. In this approach to ccdf construction, extrapolation of the tails of the distribution and interpolation between discrete points on the ccdf may not be appropriate. This is shown in Fig. A3, where a large probability (0.25) for the highest-grade sample results in the upper tail strongly influencing the univariate statistics of the ccdf, such as the mean and variance. In this example the distribution mean and variance are 1.68 ppm and 2.85 ppm$^2$ respectively. An appropriate tail extrapolation method can be used to minimise the impact of the last class on these statistics (Deutsch, Journel, 1997). However, the interpolation between the 4$^{th}$ and 5$^{th}$ sample values (0.4 - 1.8 ppm) does not account for the possibility of null-probability intervals in this part of the grade range. In the traditional IK approach this problem is solved implicitly by selecting fixed thresholds, usually based on the global cdf.

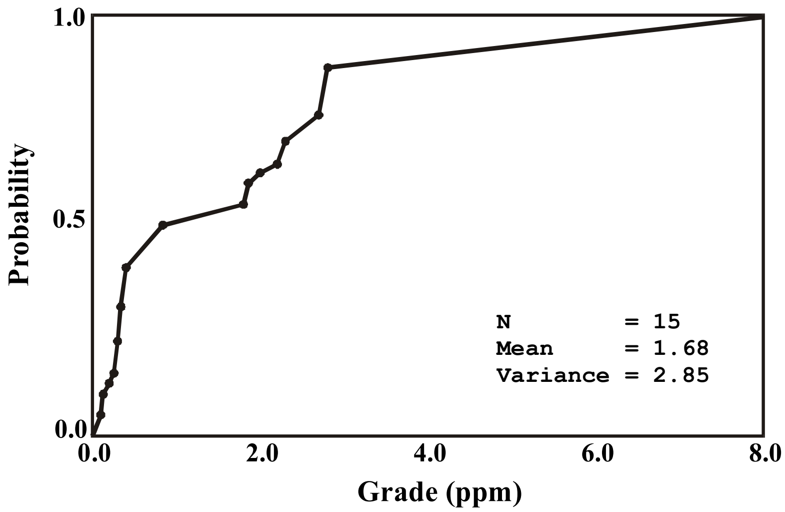


Figure A3. Cumulative distribution constructed with mid-points (Note that interpolation between discrete points on the ccdf and to the probability bounds are all linear).

As each datum contributes to the better known global cdf it may be appropriate to use the global cdf to limit the *z*-range of influence for each conditioning data $z(x_\alpha)$ on the local IK ccdf's. The idea is:

1. determine the quantile for the sample at location $x_\alpha$ from the global dataset:

$$q(x_\alpha) = F\left(A; z(x_\alpha)|(N)\right), \alpha = 1,..., N >> n$$

where $F\left(A; z(x_\alpha)\ |(N)\right)$ is the global cdf;

2. define a, say 10%, probability interval $[q_1, q_2]$ such that:

$$q_2 - q(x_\alpha) = q(x_\alpha) - q_1 = 0.05$$

3. then the $z$-range of influence for the sample at location $x_\alpha$ is:

$$z_1 = \max\left(F^{-1}(q_1); z_{\min}\right), \text{ and } z_2 = \min\left(F^{-1}(q_2); z_{\max}\right).$$

These $z$-ranges of influence may be adapted to explicitly account for discrete grade boundaries of multiple statistical populations. In practice, the $z$-ranges of influence are only used for extrapolation at the extremes of the distribution, and if there is no overlap between adjacent samples on the ccdf. Thus, interpolation and extrapolation of the ccdf is aided by the $z$-ranges of influence which provide additional points on the ccdf in the least informed (in terms of $z$-values) parts of the distribution.

Fig. A4 shows the ccdf for the previous example constructed with the $z$-ranges of influence shown in Table A1. In this diagram, linear extrapolation to the upper probability bound at 3.55 ppm does not strongly influence the univariate statistics, and more complex tail extrapolation methods are not required. In this example, the required null probability interval from 0.6 to 1.4 ppm is located such that the slope of the ccdf from 0.4-0.6 ppm equals the slope from 1.4 - 1.8 ppm.

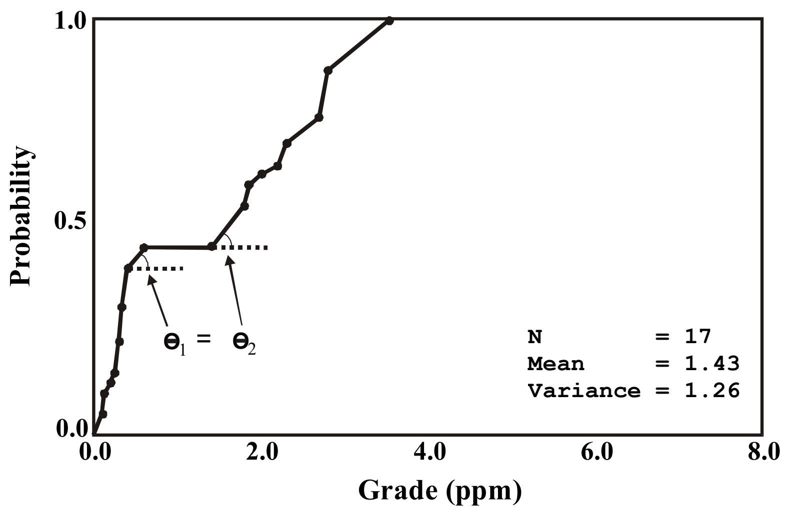


Figure A4. Cumulative distribution constructed with mid-points and z-ranges of influence (Note that interpolation between points on the ccdf and to the probability bounds are all linear).

## Appendix B – Adjusting negative kriging weights

The non-convex property of kriging does not ensure the absence of non-physical kriging weights. Solutions proposed include constraining the kriging system to deliver positive weights (*e.g.* Barnes, Johnson, 1984), or adjusting the kriging weights (*e.g.* Deutsch, 1996; Rao, Journel, 1997). Making the kriging system convex is complex and has not gained widespread popularity in the geostatistical community. Adjusting kriging weights *a posteriori* either ignores conditioning information by setting negative weights to zero (Deutsch, 1996), or artificially increases the nugget effect by adding a constant to all weights (Rao, Journel, 1997).

A practical and simple method that retains all the conditioning information and maintains the relative magnitude between kriging weights $\lambda_\alpha$ is proposed as:

1. Calculate the magnitude of the inverse of the negative kriging weights:

$$\tau = \sum_{\alpha=1}^{n} 1/\lambda_\alpha \quad \forall \quad \lambda_\alpha < 0$$

2. Correct the negative kriging weights such that they are inversely proportional to their magnitude and they sum to a small positive weight $\lambda_s$:

$$\lambda'_\alpha = \left\{ \begin{array}{ll} \dfrac{\lambda_s}{\tau \cdot \lambda_\alpha} & \forall \quad \lambda_\alpha < 0; \\ \lambda_\alpha & \text{otherwise} \end{array} \right\}$$

   The value of $\lambda_s$ is subjective, but should be such that the ratio of the largest and smallest positive weights (pre-correction) remains relatively constant.

3. Standardise the weights to sum to one:

$$\lambda''_\alpha = \frac{\lambda'_\alpha}{\sum_{\alpha=1}^{n} \lambda'_\alpha} \geq 0 \;\; \forall \alpha$$

It should be stressed that, as for any method of adjusting kriging weights, the proposed approach has no theoretical basis, and the adjusted weights vary with $\lambda_s$. However, this method ensures that only positive kriging weights are returned for all conditioning data.

Table B1 shows kriging weights assigned to 10 samples, three of which have negative weights that sum to –0.21, with the largest –0.14. The ratio between the largest and seventh largest weights is 7.40. The proposed method corrects the weights such that the largest negative weight has the smallest corrected weight. The ratios between weights are identical to the corresponding ratios between original positive kriging weights. Table B1 also includes the results using the corrections proposed by Deutsch (1996) and Rao, Journel (1997).

Table B1. Example of adjusting kriging weights for $\lambda_s = 0.05$ ($\lambda_\alpha^1$ = Deutsch method; $\lambda_\alpha^2$ = Rao, Journel method)

| $\alpha$ | $\lambda_\alpha$ | $\lambda_\alpha / \lambda_{\alpha+1}$ | $\lambda''_\alpha$ | $\lambda''_\alpha / \lambda''_{\alpha+1}$ | $\lambda^1_\alpha$ | $\lambda^1_\alpha / \lambda^1_{\alpha+1}$ | $\lambda^2_\alpha$ | $\lambda^2_\alpha / \lambda^2_{\alpha+1}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.370 | 1.762 | 0.294 | 1.762 | 0.306 | 1.762 | 0.213 | 1.457 |
| 2 | 0.210 | 1.105 | 0.167 | 1.105 | 0.174 | 1.105 | 0.146 | 1.061 |
| 3 | 0.190 | 1.188 | 0.151 | 1.188 | 0.157 | 1.188 | 0.138 | 1.100 |
| 4 | 0.160 | 1.067 | 0.127 | 1.067 | 0.132 | 1.067 | 0.125 | 1.034 |
| 5 | 0.150 | 1.875 | 0.119 | 1.875 | 0.124 | 1.875 | 0.121 | 1.318 |
| 6 | 0.080 | 1.600 | 0.063 | 1.600 | 0.066 | 1.600 | 0.092 | 1.158 |
| 7 | 0.050 | - | 0.040 | 1.543 | 0.041 | - | 0.079 | 1.583 |
| 8 | -0.020 | - | 0.026 | 2.500 | 0.000 | - | 0.050 | 1.333 |
| 9 | -0.050 | - | 0.010 | 2.800 | 0.000 | - | 0.038 | - |
| 10 | -0.140 | - | 0.004 | - | 0.000 | - | 0.000 | - |
| | | $\lambda_1/\lambda_7 = 7.40$ | | $\lambda''_1/\lambda''_7 = 7.35$ | | $\lambda^1_1/\lambda^1_7 = 7.46$ | | $\lambda^2_1/\lambda^2_7 = 2.70$ |

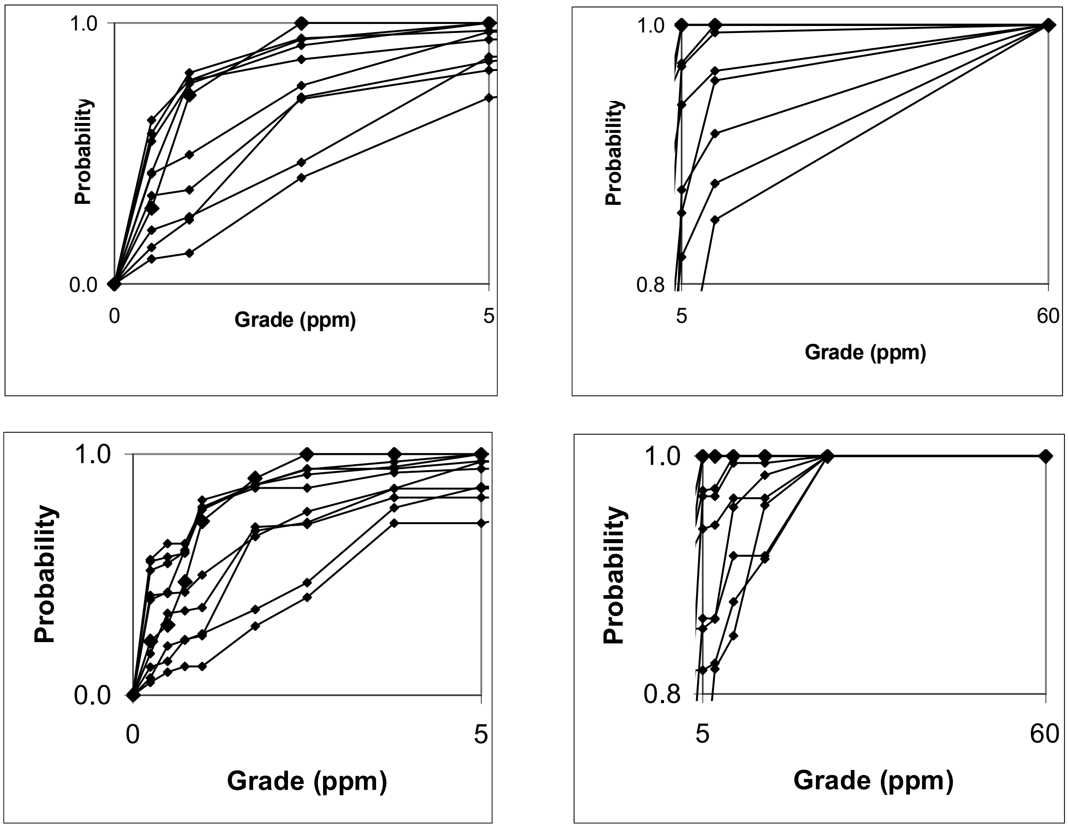


Figure 1. Conditional cumulative distributions for 10 locations estimated by IK: (top) 5 thresholds corresponding to the 10th, 30th, 50th, 70th, and 90th percentiles; (bottom) 12 thresholds corresponding to the deciles and 95th, 97.5th, and 99th percentiles.

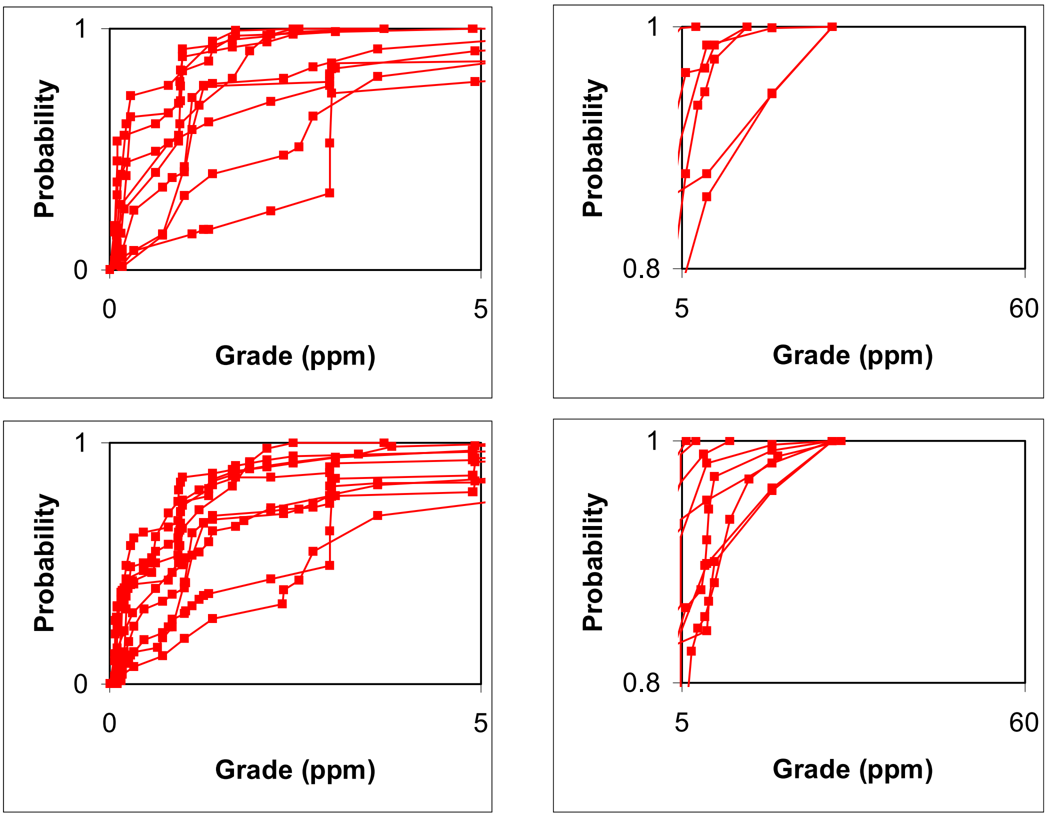


Figure 2. Conditional cumulative distributions estimated by the proposed IK method for the same 10 locations shown in Fig. 1: (top) 12 samples; (bottom) 24 samples.

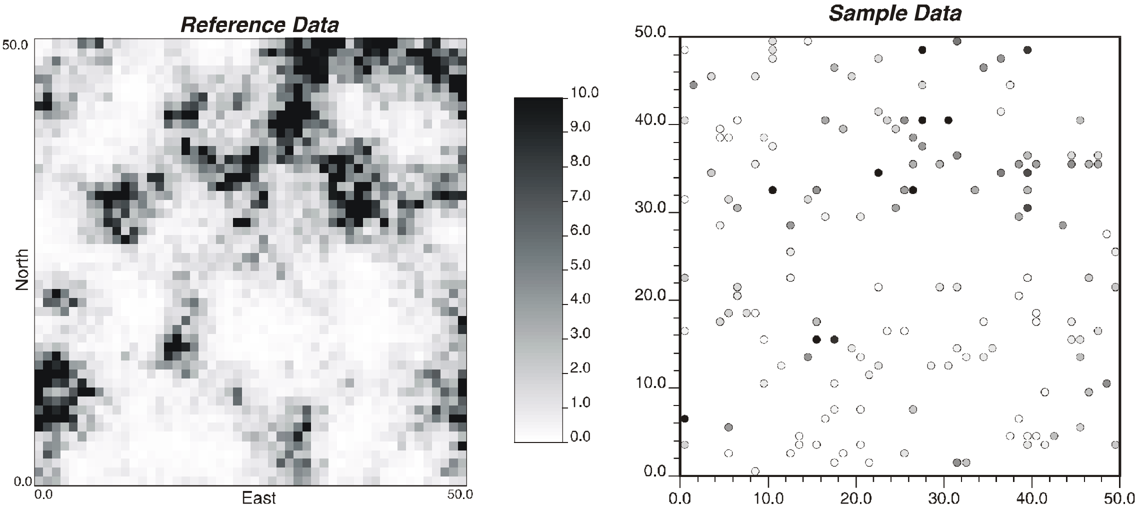


Figure 3. Reference and first sample dataset.

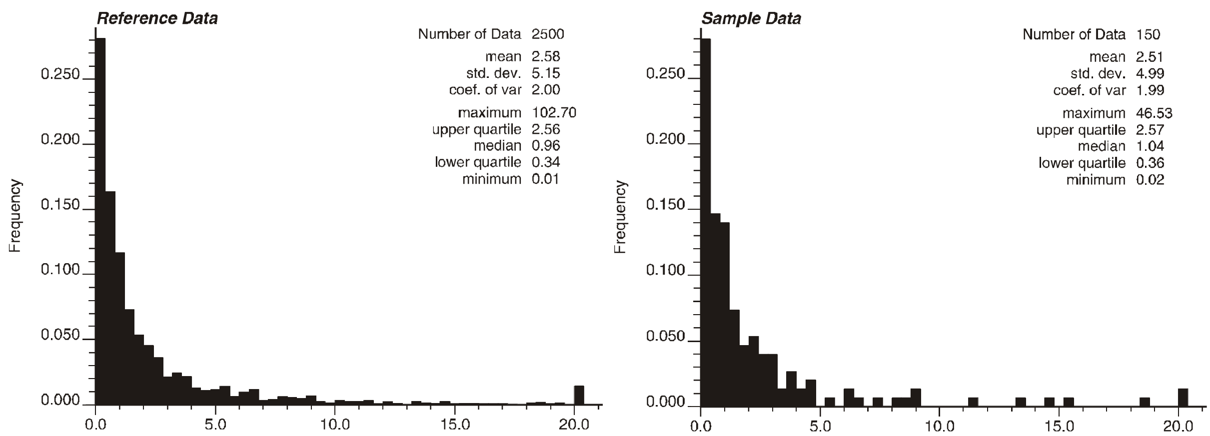


Figure 4. Reference and first sample dataset statistics.

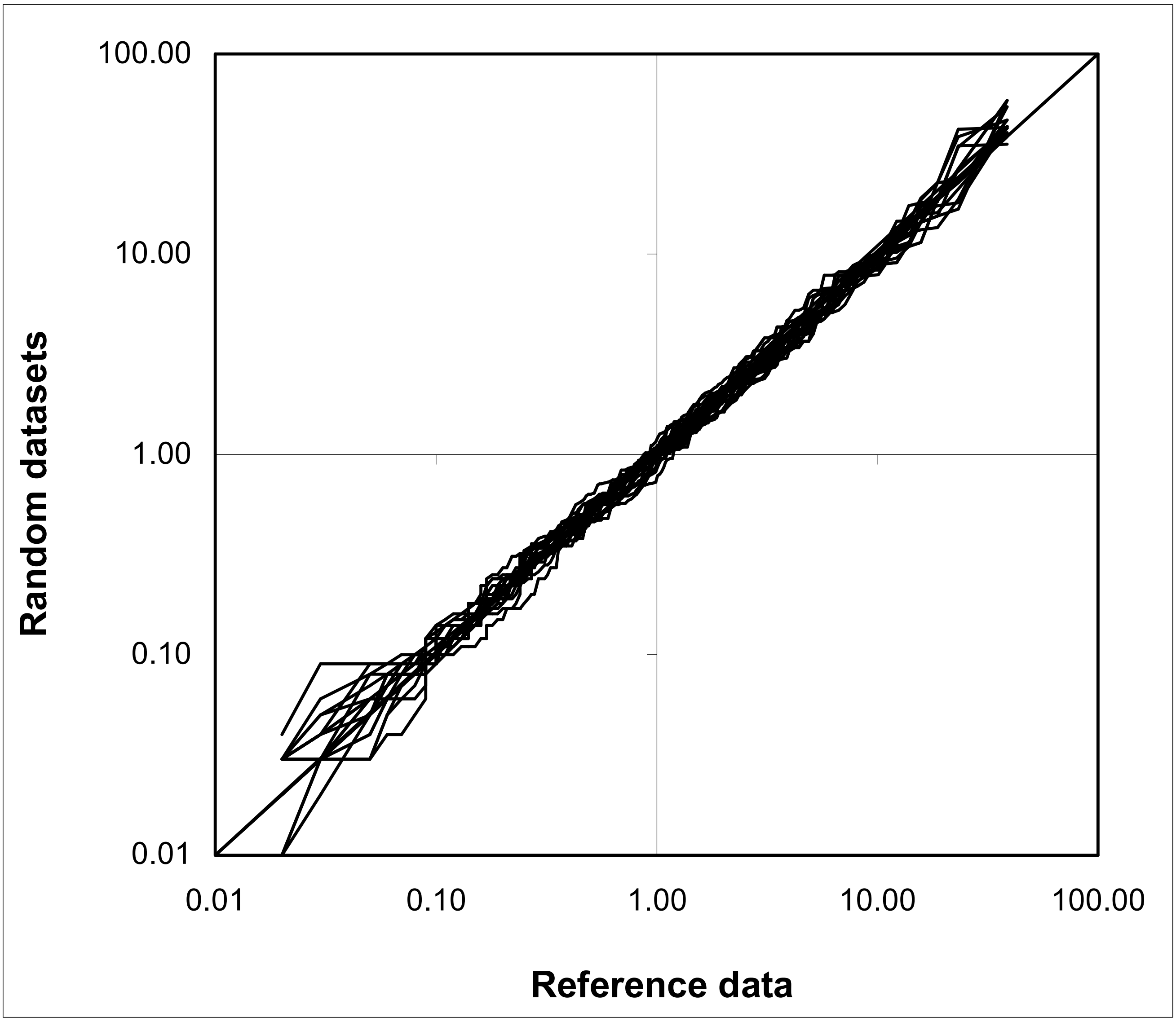


Figure 5. Q-Q plots of reference and 20 sample datasets.

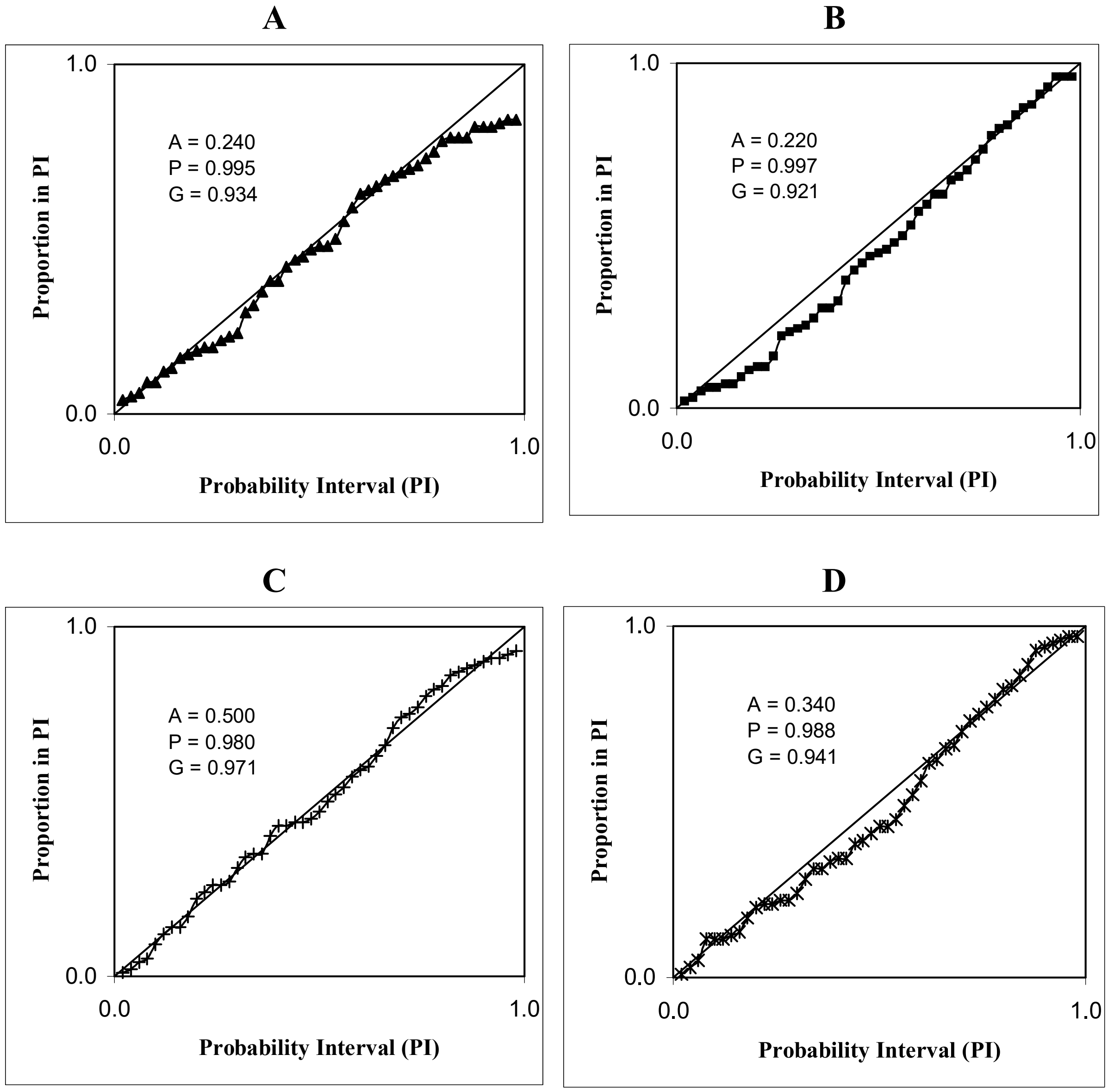


Figure 6. Accuracy plots for the IK models using the first sample dataset: (A) IK-1; (B) IK-2; (C) DTIK-1, and (D) DTIK-2.

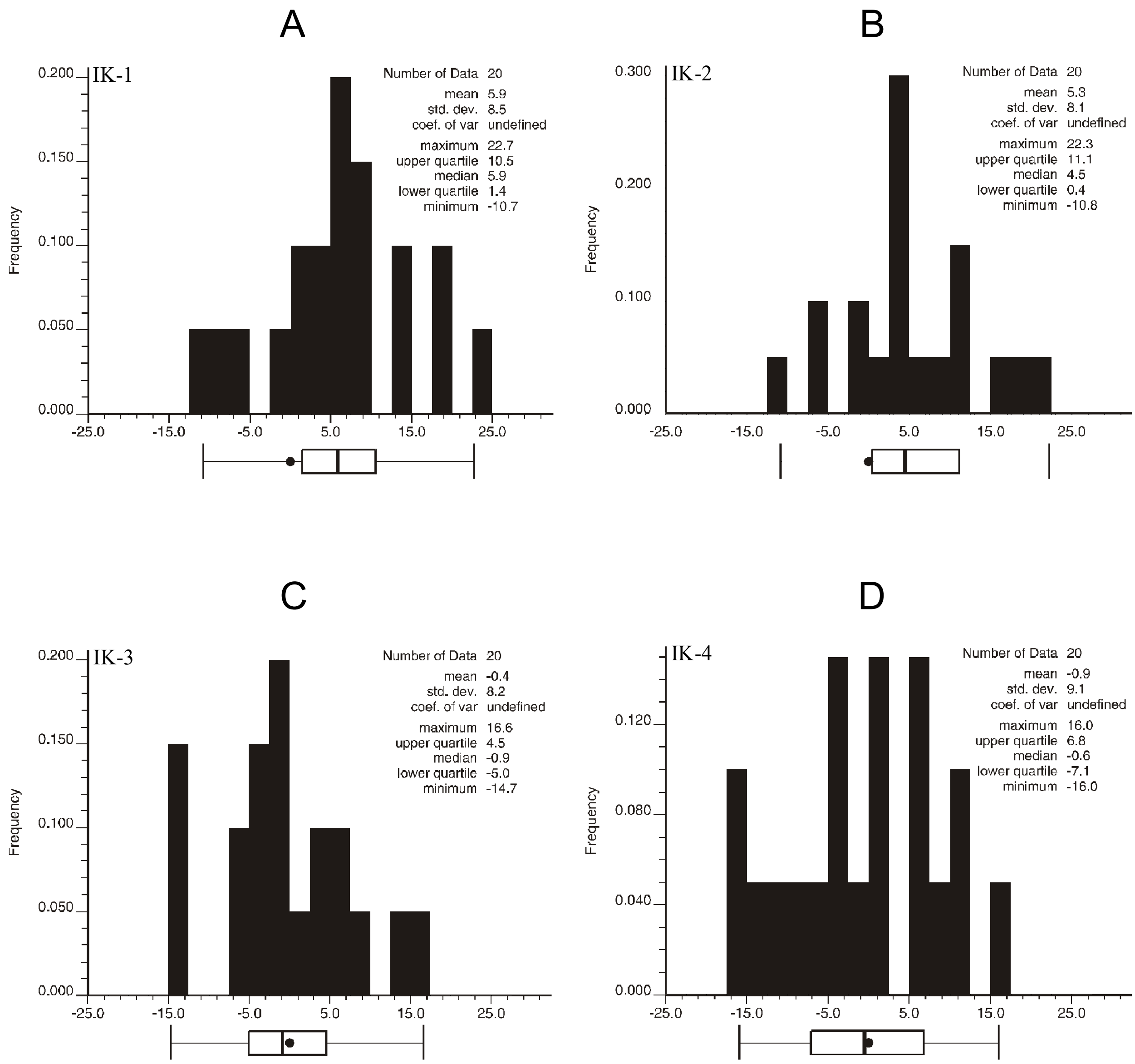


Figure 7. Histograms of recoverable tonnage error (%) at a 1 x 1 support and 2.5 units cut-off: (A) IK-1 with a hyperbolic model upper tail extrapolation (ω=1.5); (B) IK-2 with a linear upper tail extrapolation; (C) DTIK-1 with a linear upper tail extrapolation, and (D) DTIK-2 with a linear upper tail extrapolation (Note that the error is relative to the true recoverable tonnage).

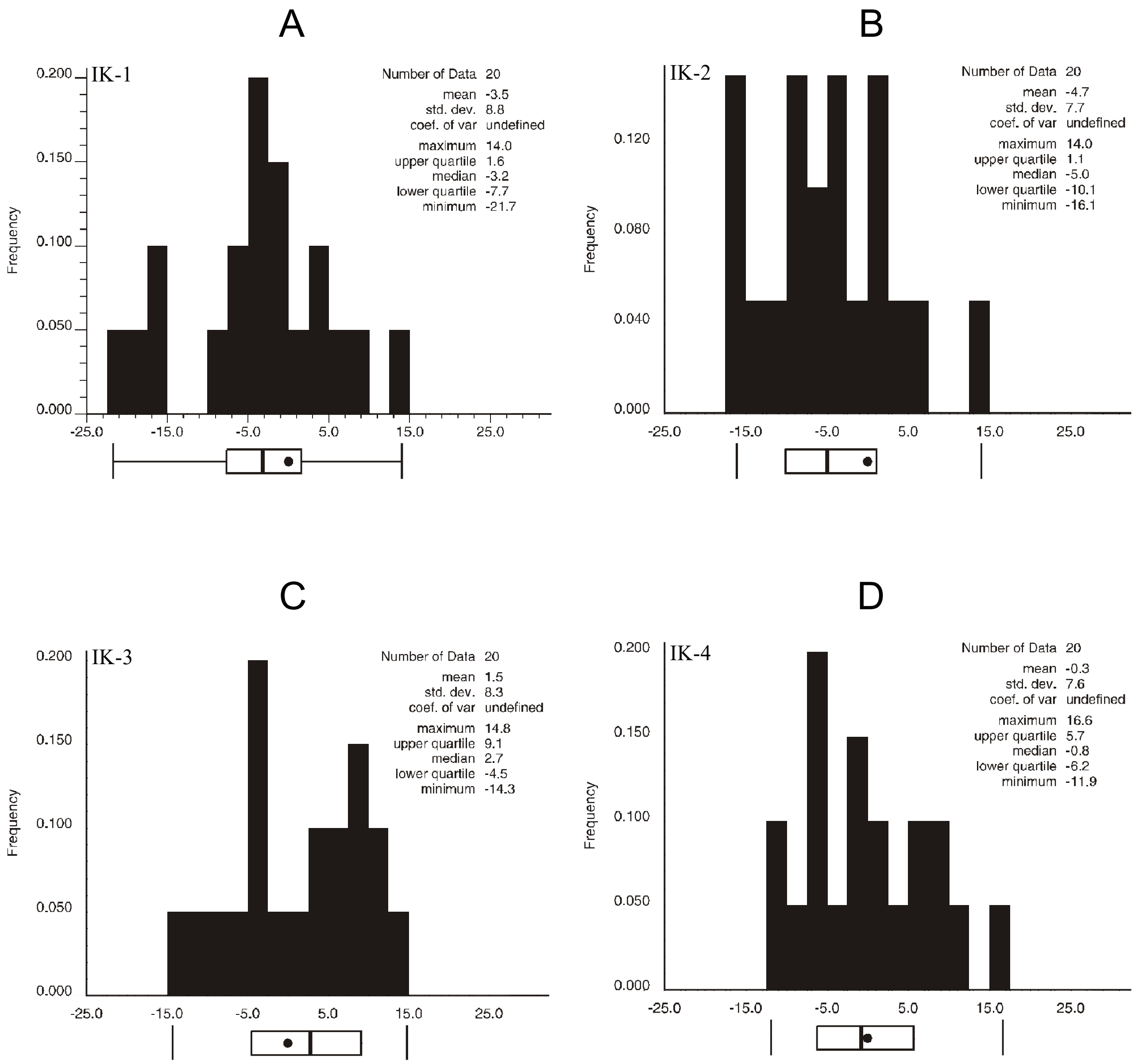


Figure 8. Histograms of recoverable metal error (%) at a 1 x 1 support and 2.5 units cut-off: (A) IK-1 with a hyperbolic model upper tail extrapolation (ω=1.5); (B) IK-2 with a linear upper tail extrapolation; (C) DTIK-1 with a linear upper tail extrapolation, and (D) DTIK-2 with a linear upper tail extrapolation (Note that the error is relative to the true recoverable metal).

Table 1. Indicator variogram parameters (after Deutsch, Journel, 1997).

| $z_k$ | $F(A;z_k)$ | $C_0(z_k)$ | $C_1(z_k)$ | $a_1(z_k)$ | $C_2(z_k)$ | $a_2(z_k)$ |
|---|---|---|---|---|---|---|
| 0.159 | 0.1 | 0.0 | 0.5 | 11.0 | 0.5 | 30.0 |
| 0.278 | 0.2 | 0.0 | 1.0 | 11.0 | 0.0 | - |
| 0.515 | 0.3 | 0.0 | 1.0 | 11.0 | 0.0 | - |
| 0.918 | 0.4 | 0.0 | 1.0 | 11.0 | 0.0 | - |
| 1.211 | 0.5 | 0.0 | 1.0 | 11.0 | 0.0 | - |
| 1.708 | 0.6 | 0.0 | 1.0 | 11.0 | 0.0 | - |
| 2.325 | 0.7 | 0.0 | 1.0 | 11.0 | 0.0 | - |
| 3.329 | 0.8 | 0.4 | 0.6 | 11.0 | 0.0 | - |
| 5.384 | 0.9 | 0.9 | 0.9 | 11.0 | 0.0 | - |

Table 2. Statistics of various IK implementations (Note: statistics are the average of the 20 models).

| Statistic | True | IK-1 | IK-2 | DTIK-1 | DTIK-2 |
|---|---|---|---|---|---|
| Average block grade | 2.58 | 2.49 | 2.47 | 2.62 | 2.58 |
| Variance of block mean grades | 9.24 | 4.00 | 4.12 | 5.87 | 3.64 |
| Average co-efficient of variation of block ccdf's | 0.94 | 1.49 | 1.56 | 1.19 | 1.48 |